SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

2015

HSC ASSESSMENT TASK # 3

Mathematics Extension 1

General Instructions

- Reading time - 5 minutes
- Working time - 90 Minutes
- Write using black or blue pen. Black pen is preferred.
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- In Questions 8 - 10, show **ALL** relevant mathematical reasoning and/or calculations.
- Marks may **NOT** be awarded for untidy or badly arranged work.
- Answer in simplest **EXACT** form unless otherwise instructed.

Total marks - 60

Section I Pages 2 – 5

(7 marks)

- Attempt Questions 1 – 7 on the Multiple Choice answer sheet provided.
- Allow about 10 minutes for this section.

Section II Pages 6 – 15

(53 marks)

- Attempt Questions 8 – 10.
- Start a new answer booklet for each question.
- Allow about 1 hours and 20 minutes for this section.

Examiner: *J. M.*

This is an assessment task only and does not necessarily reflect the content of the Higher School Certificate.

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Section I

7 marks
Attempt Questions 1–7
Allow about 10 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–7.

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1. The exact value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\cos^{-1}\left(\frac{1}{2}\right)$ is

(A) 0.822

(B) $\frac{7\pi}{24}$

(C) $\frac{\pi^2}{24}$

(D) $\frac{\pi^2}{12}$

2. At a dinner party, the host, hostess and their six guests sit at a round table. In how many ways can they be arranged if the host and hostess are separated?

(A) 5040

(B) 3600

(C) 1440

(D) 720

3. Which one of these functions has an inverse relation that is not a function?

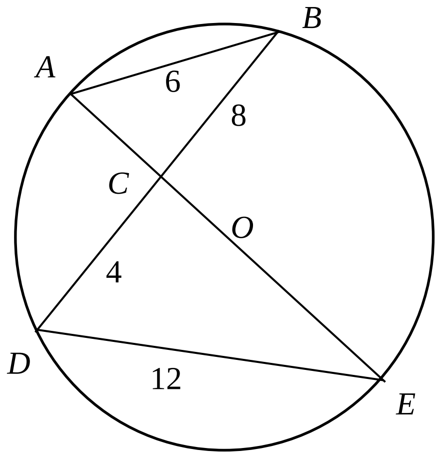
(A) $y = x^3$

(B) $y = \ln x$

(C) $y = \sqrt{x}$

(D) $y = |x|$

4. In the figure shown, four chords are drawn so that two intersects at point *C* and chord *AE* is the diameter of the circle. If *AB* = 6, *BC* = 8, *CD* = 4 and *DE* = 12, then what is the area of the circle?



(A) $96\pi\ \text{u}^2$

(B) $81\pi\ \text{u}^2$

(C) $64\pi\ \text{u}^2$

(D) $27\pi\ \text{u}^2$

5. Using the substitution of $u = 2x - 3$, the integral of $\int \frac{1}{(2x-3)^2+1}\,dx$ is equal to

(A) $\frac{1}{2x-3}\tan^{-1}(2x-3)+C$

(B) $\tan^{-1}(2x-3)+C$

(C) $\frac{1}{2}\tan^{-1}(2x-3)+C$

(D) $-\frac{1}{3}\tan^{-1}(2x-3)+C$

6. The radius of a sphere is increasing at a rate of 3 cm/min. When the radius is 8 cm, the rate of increase, in cm^3/min, of the volume of the sphere is:

(A) $85\frac{1}{3}\pi$

(B) 265π

(C) $682\frac{2}{3}\pi$

(D) 768π

7. Two particles start at the origin and move along the *x* axis. For $0 \le t \le 10$, their respective position functions are given by $x_1 = \sin t$ and $x_2 = e^{-2t} - 1$. For how many values of *t* do the particles have the same velocity?

(A) One

(B) Two

(C) Three

(D) Four

End of Section I

Section II

53 marks
Attempt Questions 8 – 10
Allow about 1 hour and 20 minutes for this section.

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 8 – 10, your responses should include ALL relevant mathematical reasoning and/or calculations.

Question 8 (17 marks) Start a new writing booklet for each question.

(a) Find the gradient of the tangent to the curve $y = \sin^{-1}\left(\frac{x}{2}\right)$ at the point where $x = 1$. **2**

(b) From a group of 7 men and 6 women, five people are to be selected to form a committee. How many committees can be formed if:

(i) There are at least 3 men on the committee? **2**

(ii) One particular man must be on the committee and one particular woman must not be on the committee? **2**

(iii) One particular man and one woman refuse to be on the committee together? **2**

(c) Use mathematical induction to show that for all integers $n \geq 1$, **3**

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}.$$

Question 8 continues on the next page

(d) By using the substitution of $u = 27 - x^3$, find the volume of the solid formed when the region bounded by the x axis and the curve $y = x\left(27 - x^3\right)^2$ between $x = 0$ and $x = 2$ is rotated about the x axis. **3**

(e) A resting adult's breathing cycle is 5 seconds long. For time t seconds, $0 \le t \le 2.5$, air is taken into the lungs. For $2.5 \le t \le 5$ air is expelled from the lungs. **3**

The rate, R litres/second, at which air is taken in or expelled from the lungs can be modelled on the equation.

$$R = \frac{1}{2}\sin\left(\frac{2\pi}{5}t\right)$$

How many litres of air does a resting adult take into their lungs during one breathing cycle?

End of Question 8

Question 9 (18 marks) Start a new writing booklet for each question.

(a) At 3 pm in a school playground, a melted chocolate bar is found to be at a temperature of $x°C$. The packaging on the chocolate bar says it will begin to melt at $16°C$. The temperature, $T°C$, of the bar t minutes after 3 pm varies according to the differential equation

$$\frac{dT}{dt} = \frac{1}{60}\ln(1.6)\,(A - T),$$

where A is a constant.

(i) Show that for any constant, B, a solution to the differential equation is **2**

$$T = A + Be^{-\frac{1}{60}\ln(1.6)\,t}$$

(ii) After an hour, the chocolate bar increases in temperature by $3.75°C$. Given the chocolate bar started to melt at 2 pm, find x and the limiting temperature of the bar. Assume the temperature of the day is constant. **3**

(b) Use the substitution $u = e^x$ to evaluate $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx$. **3**

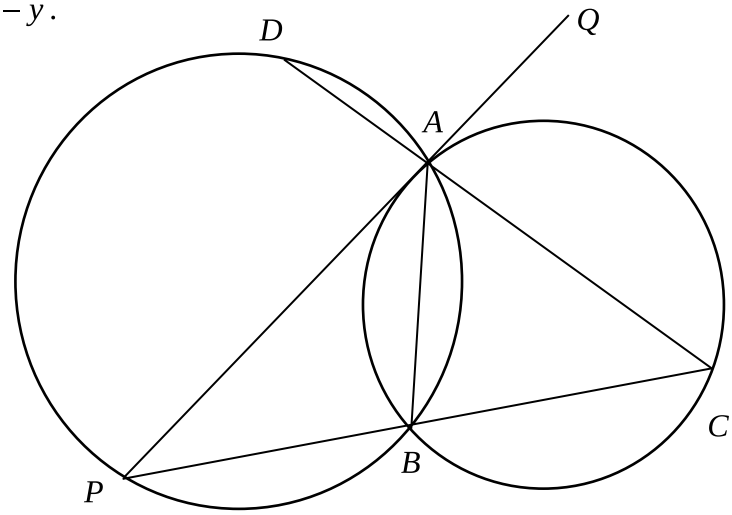
(c) Given $f(x) = 2\cos^{-1}\left(\frac{x}{\sqrt{2}}\right) - \sin^{-1}\left(1 - x^2\right)$. Show that $f'(x) = 0$ for $x > 0$. Hence sketch $f(x)$ for $x \geq 0$. **4**

Question 9 continues on the next page

(d) Prove by induction that for every integer $n \geq 1$. **3**

$$\frac{d^n}{dx^n}\left(xe^{2x}\right) = 2^{n-1}\left(2x+n\right)e^{2x}$$

(e) The diagram below, shows two unequal circles intersecting at A and B. the tangent to the smaller circle at A cuts the larger circle at P, PB cuts the smaller circle at C and CA cuts the larger circle at D. If $\angle QAC = x$ and $\angle PAB = y$, show, giving reasons that $\angle BDA = x - y$. **3**



End of Question 9

Question 10 (18 marks) Start a new writing booklet for each question.

(a) A car braked with a constant deceleration of 16 m/s^2, producing skid marks measuring 200 m before coming to a stop. How fast was the car traveling when the brakes were first applied? **4**

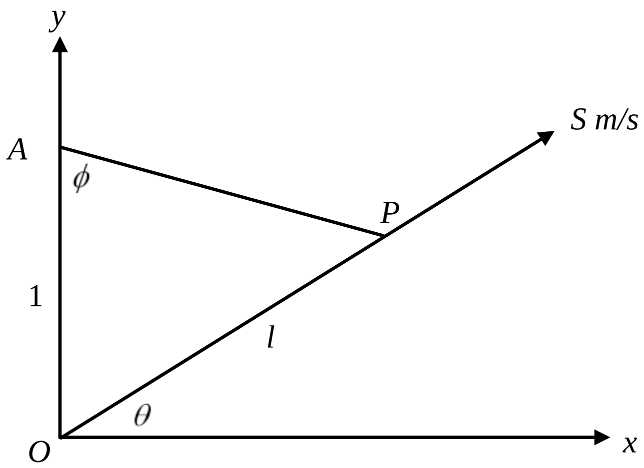
(b) Consider the function $f(x) = 3x\tan^{-1}(2x)$.

(i) Write down the range of $f(x)$. **2**

(ii) Show that $f'(x) = 3\tan^{-1}(2x) + \dfrac{6x}{1+4x^2}$. **2**

(iii) Hence, evaluate the area enclosed by the $g(x) = \tan^{-1}(2x)$, the x axis and the lines $x = \dfrac{1}{2}$ and $x = \dfrac{\sqrt{3}}{2}$. **3**

(c) A point A lies on the y axis, 1 unit away from the origin in the positive direction. Another point P lies on the line $y = x\tan\theta$, where $0° \le \theta \le 90°$, and travels along the line with some speed S m/s. Let l be the distance from P to the origin, and let $\angle OAP$ be ϕ.



(i) Prove that $l = \dfrac{\sin\phi}{\cos(\phi-\theta)}$. **2**

(ii) Show that when $\phi = 2\theta$,

(1) $\dot{\phi} = S\cos\theta$. **3**

(2) $\ddot{\phi} = -\dot{\phi} \times S \times l$. **2**

End of Exam

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad n \neq 0; \quad \text{if } n < 0.$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right),$$

Note: $\ln x = \log_e x \qquad x > 0$

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HSC Task #3

Mathematics Extension 1

Suggested Solutions & Markers' Comments

QUESTION	Marker
1 – 7	–
8	BK
9	RB
10	DH

Multiple Choice Answers

1. D
2. B
3. D
4. B
5. C
6. D
7. C

Solutions for Multiple Choice Y12 Ext 1 Task 3 2015

The mean score for this question was 5.93/7

Q1

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\cos^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4} \times \frac{\pi}{3}$$

$$= \frac{\pi^2}{12}$$

D

A	5
B	1
C	5
D	154

Q2

Host

5 6

1 5

2 3 4

$\therefore\ 5 \times 6! = 3600$

B

A	3
B	145
C	13
D	4

Q3

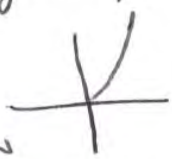
$y = x^3 \Rightarrow y = x^{1/3}$

FUNCTION

$y = \ln x \Rightarrow y = e^x$

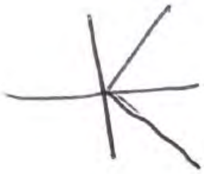
FUNCTION

$y = \sqrt{x} \Rightarrow y = x^2,\ x \geq 0$



FUNCTION

$y = |x| \Rightarrow$



NOT A FUNCTION

D

A	3
B	3
C	10
D	149

Q4

$\Delta ABC \parallel\!\mid \Delta DEC$

$\frac{AC}{4} = \frac{6}{12} \Rightarrow AC = 2$

$\frac{CE}{8} = \frac{12}{6} \Rightarrow CE = 16$

$\therefore AE = 18$

$\therefore OE = 9$

$\therefore \text{Area} = \pi . 9^2$

$= 81\pi \ u^2$

B

A	7
B	133
C	21
D	3

Q5

5. $\int \frac{1}{(2x-3)^2+1} dx$ $\qquad u = 2x-3$

$\therefore du = 2dx$

$= \int \frac{1}{u^2+1} \left(\frac{1}{2}\right) du$

$= \frac{1}{2} \tan^{-1} u + c$

$= \frac{1}{2} \tan^{-1}(2x-3) + c$

C

A	1
B	4
C	160
D	0

Q6

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 . \frac{dr}{dt}$

$= 4\pi . 64 . 3$

$= 768\pi \ cm^3/min$

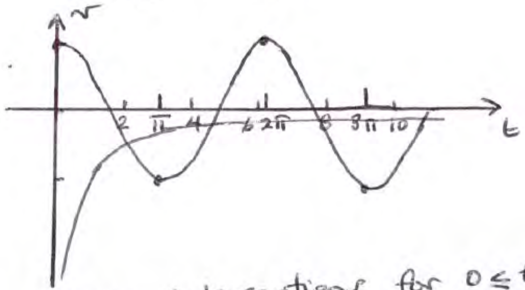
D

A	1
B	2
C	2
D	160

Q7

7. $r_1 = \cos t$

$r_2 = -2e^{-2t}$



Three intersections for $0 \leq t \leq 10$

C

A	36
B	42
C	67
D	20

Ext 1 – Task 3

Q8(a) $y = \sin^{-1}\left(\frac{x}{2}\right)$

$y' = \frac{1}{\sqrt{4-x^2}}$ ✓

(2)

$y'(1) = \frac{1}{\sqrt{3}}$ = gradient of tangent at $x=1$. ✓

(b) (i) At least 3M ⇒ 3 or 4 or 5 men

⇒ 3M, 2W or 4M, 1W or 5M

$= {}^7C_3 \times {}^6C_2 + {}^7C_4 \times {}^6C_1 + {}^7C_5$

$= 525 + 210 + 21$

= 756 committees.

* Some students used the complement but some forgot to include the possibility of '0 men'. ie. they needed 1 − (0 men or 1 man or 2 men).

(ii) One particular man on and 1 particular woman not on

$= 1 \times {}^{11}C_4$ = 330 committees.

or 1M, 4W or 2M, 3W or 3M, 2W or 4M, 1W
where put 1 man on + select rest from remaining 6M. Select from 5W.

$= 1 \times {}^5C_4 + 1 \times {}^6C_1 \times {}^5C_3 + 1 \times {}^6C_2 \times {}^5C_2$
$+ 1 \times {}^6C_3 \times {}^5C_1 + 1 \times {}^6C_4$

$= 5 + 60 + 150 + 100 + 15$

= 330 committees

8(b)(iii) 1 particular man <u>on</u> and 1 particular woman <u>not</u> on

= Total ways − no. ways together

$= {}^{13}C_5 - 1 \times 1 \times {}^{11}C_3$ ✓

$= 1287 - 165$ ✓ (2)

$= \underline{1122 \text{ committees}}$.

* Some students chose from 7 men when the particular man was <u>on</u> committee.
* Some students incorrectly used Permutations

8(c) $P(n): 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+\dots+n} = \frac{2n}{n+1}$

* This question should have had $\underline{n \geq 1}$.

$P(1)$: LHS = 1, RHS $= \frac{2\times 1}{2} = 1$ ∴ $P(1)$ true.

Assume $P(k)$ true

ie $P(k): 1 + \frac{1}{1+2} + \dots + \frac{1}{1+2+\dots+k} = \frac{2k}{k+1}$ ✓

Let $n = k+1$. Must show $P(k+1)$ true

ie show $1 + \frac{1}{1+2} + \dots + \frac{1}{1+2+\dots+k} + \frac{1}{1+2+\dots+k+k+1} = \frac{2(k+1)}{k+2}$

(3)

$$LHS = \left(1 + \frac{1}{1+2} + \dots + \frac{1}{1+2+\dots+k}\right) + \frac{1}{1+2+\dots+k+k+1}$$

$$= \frac{2k}{k+1} + \frac{1}{1+2+\dots+k+k+1}$$ ✓ from induction step

Now sum of 1st $(k+1)$ numbers $= \frac{(k+1)(k+2)}{2}$

$$\therefore LHS = \frac{2k}{k+1} + \frac{1}{\frac{(k+1)(k+2)}{2}} = \frac{2k}{k+1} + \frac{2}{(k+1)(k+2)}$$

$$= \frac{2k^2+4k+2}{(k+1)(k+2)} = \frac{2(k+1)^2}{(k+1)(k+2)} = \frac{2(k+1)}{k+2} = RHS$$ ✓

∴ true for $n = k+1$.

∴ by Principle of Mathematical Induction true $\forall n \in \mathbb{Z}^+$

8(d) $V = \pi \int y^2 dx$

$= \pi \int_0^2 x^2 (27 - x^3)^4 dx$

Let $u = 27 - x^3$

$du = -3x^2 dx$

Also when $x = 0, u = 27$

$x = 2, u = 19$

$V = -\pi \int_{19}^{27} x^2 u^4 \frac{du}{-3x^2}$

$= \frac{\pi}{3} \int_{19}^{27} u^4 du$

$= \frac{\pi}{3} \left[\frac{u^5}{5} \right]_{19}^{27}$

$= \frac{\pi}{3} \left[\frac{27^5 - 19^5}{5} \right]$

* Some students left answer like this and lost ½ m.

$= \frac{\pi \times 11872808}{15}$

$= \frac{11872808 \pi}{15} \ u^3.$

* Students lost ½ m for rounding answer
* Some students forgot to square y for Volume.

8(e) $R = \frac{1}{2} \sin \left(\frac{2\pi}{5} t \right)$

$\therefore$ Air in $= \int_0^{2.5} \left(\frac{1}{2} \sin \frac{2\pi}{5} t \right) dt$ ✓

$= \frac{-5}{4\pi} \left[\cos \frac{2\pi}{t} \right]_0^{2.5}$ ✓ $= \frac{-5}{4\pi} \left[\cos \pi - \cos 0 \right]$

$= \frac{-5}{4\pi} (-1 - 1) = \frac{10}{4\pi}$

$= \frac{5}{2\pi}$ litres ✓

(3)

* ½ m lost if rounded
* Some students found indefinite integral but some forgot to add constant, C.

9 (a) $\frac{dT}{dt} = \frac{1}{60}\ln(1.6)(A-T)$

Reasonably well answered but for 2 marks, all lines of working needed to be shown. Regardless of the method(s) used, students should not skip non-obvious lines of working.

(i) $T = A + Be^{-\frac{1}{60}\ln(1.6)t}$

$\frac{dT}{dt} = 0 + -\frac{1}{60}\ln 1.6\, Be^{-\frac{1}{60}\ln(1.6)t}$

now if $T = A + Be^{-\frac{1}{60}\ln(1.6)t}$

$-Be^{-\frac{1}{60}\ln(1.6)t} = A - T$

so $\frac{dT}{dt} = \frac{1}{60}\ln 1.6 \times -Be^{-\frac{1}{60}\ln(1.6)t}$

$= \frac{1}{60}\ln 1.6\,(A-T)$ (2)

(ii) OR $\frac{dT}{dt} = \frac{1}{60}\ln(1.6)(A-T)$

$\int \frac{dT}{(A-T)} = \int \frac{1}{60}\ln(1.6)\,dt$

$-\ln(A-T) = \frac{1}{60}\ln(1.6)t + c$

$\ln(A-T) = -\left(\frac{1}{60}\ln(1.6)t + c\right)$

$e^{\ln(A-T)} = e^{-\left(\frac{1}{60}\ln(1.6)t + c\right)}$

$A - T = e^{-\frac{1}{60}\ln(1.6)t - c} = e^{-\frac{1}{60}\ln(1.6)t} \times e^{-c}$

Let $B = e^{-c}$

$A - T = B.e^{-\frac{1}{60}\ln(1.6)t}$

$\therefore T = A + Be^{-\frac{1}{60}\ln(1.6)t}$

then $T = A - e^{-\frac{1}{60}\ln(1.6)t + c}$

$T = A + Be^{-\frac{1}{60}\ln(1.6)t}$

when $t = -60$ $T = 16$ $16 = A + Be^{-\frac{1}{60}\ln(1.6)\times -60}$

$16 = A + Be^{\ln 1.6}$ ①

when $t = 0$, $T = x$. $x = A + B$ ②

when $t = 60$ $T = x + 3.75$ $x + 3.75 = A + Be^{-\frac{1}{60}\ln(1.6)\times 60}$

$x + 3.75 = A + Be^{-\ln(1.6)}$ ③

② & ③ $\cancel{A} + B + 3.75 = \cancel{A} + Be^{-\ln(1.6)}$

$B + 3.75 = B \times 0.625$

$3.75 = -B + 0.625B$

$3.75 = B(-1 + 0.625)$

$B = \frac{3.75}{-0.375} = \boxed{-10}$

now

$e^{\ln x} = x$

$e^{\ln 1.6^{-1}} = 0.625$

so ① $16 = A - 10e^{\ln 1.6}$

$16 = A - 10 \times 1.6$

$16 = A - 16$

$A = \boxed{32}$

Badly answered - Conclusion; too much possible confusing data. Marker marked very liberally.

Thus in ② $x = A + B = 32 - 10 = 22$

②

Limiting temp? $T = 32 - 10e^{-\frac{1}{60}\ln(1.6)t}$

$= 32 - \frac{10}{e^{\frac{1}{60}\ln(1.6)t}}$ (as t becomes large)

$\rightarrow 32^{0}$.

①

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9 (b) $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx$ $\qquad e^{2x} = (e^x)^2$

let $u = e^x$

$\frac{du}{dx} = e^x$

$du = e^x\,dx$

$$\int \frac{du}{1+u^2} = \tan^{-1}u \Rightarrow \tan^{-1}e^x\Big]_0^{\ln\sqrt{3}} \quad (1)$$

$= \tan^{-1}e^{\ln\sqrt{3}} - \tan^{-1}e^0$ $\qquad$ uses $e^{\ln x} = x$

$= \tan^{-1}\sqrt{3} - \tan^{-1}1 \quad (1)$

$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi - 3\pi}{12} = \frac{\pi}{12} \quad (1)$

well answered but a small number of students decided that $\int \frac{1}{1+u^2}\,du$ was a ln answer

no!

9 (c) $f(x) = 2\cos^{-1}\left(\frac{x}{\sqrt{2}}\right) - \sin^{-1}(1-x^2)$

$$f'(x) = \frac{2 \cdot -1}{\sqrt{1-\frac{x^2}{2}}} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{1-(1-x^2)^2}} \times -2x$$

$$= -\frac{2}{\sqrt{2}} \frac{1}{\sqrt{\frac{2-x^2}{2}}} + \frac{2x}{\sqrt{1-(1-2x^2+x^4)}} \quad (1)$$

$$= -\frac{2}{\sqrt{2}} \cdot \frac{1}{\frac{\sqrt{2-x^2}}{\sqrt{2}}} + \frac{2x}{\sqrt{1-1+2x^2-x^4}}$$

$$= -\frac{2}{\sqrt{2}} \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2-x^2}} \qquad \frac{2x}{\sqrt{x^2(2-x^2)}}$$

$$= \frac{-2}{\sqrt{2-x^2}} + \qquad = \frac{2x}{|x|\sqrt{2-x^2}}$$

reasonably attempted but many students tried to fudge both parts of the calculation

$$f'(x) = \frac{-2}{\sqrt{2-x^2}} + \frac{2}{\sqrt{2-x^2}} = 0 \quad (1)$$

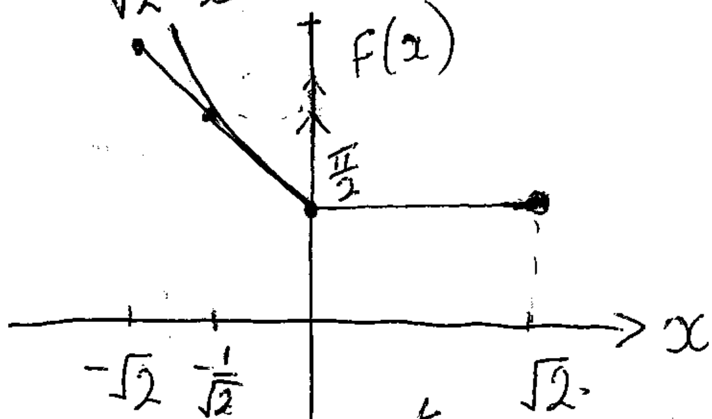


diagram: domain

$$-1 \le \frac{x}{\sqrt{2}} \le 1$$
$$-\sqrt{2} \le x \le \sqrt{2}$$

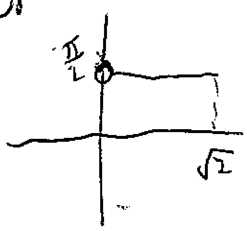
and $-1 \le 1-x^2 \le 1$ (1)

$$-2 \le -x^2 \le 0$$

$-2 \le -x^2$ and $-x^2 \le 0$
$2 \ge x^2$ $\quad x^2 \ge 0$ ✓
$x^2 \le 2 \Rightarrow -\sqrt{2} \le x \le \sqrt{2}$

Many students forgot the domain or took it as $-1 \le x \le 1$.

(2)



(d) $\frac{d^n}{dx^n}(xe^{2x}) = 2^{n-1}(2x+n)e^{2x}$

n integer ≥ 1

Step 1 Let $n=1$, So $\frac{d}{dx}(xe^{2x}) = xe^{2x} \times 2 + e^{2x} \times 1$

$= e^{2x}(2x+1)$.

RHS. Let $n=1$, $2^0(2x+1)e^{2x}$

$= e^{2x}(2x+1)$

$n=1$ is true ①

Many students did not show LHS = RHS when $n=1$. Needs to be clearly shown.

Step 2 Let there be a value of $n=k$ where k is an integer ≥ 1 then

$$\frac{d^k}{dx^k}(xe^{2x}) = 2^{k-1}(2x+k)e^{2x}$$

and we must prove for $n=k+1$ that

$$\frac{d^{k+1}}{dx^{k+1}}(xe^{2x}) = 2^{k+1-1}(2x+k+1)e^{2x}$$

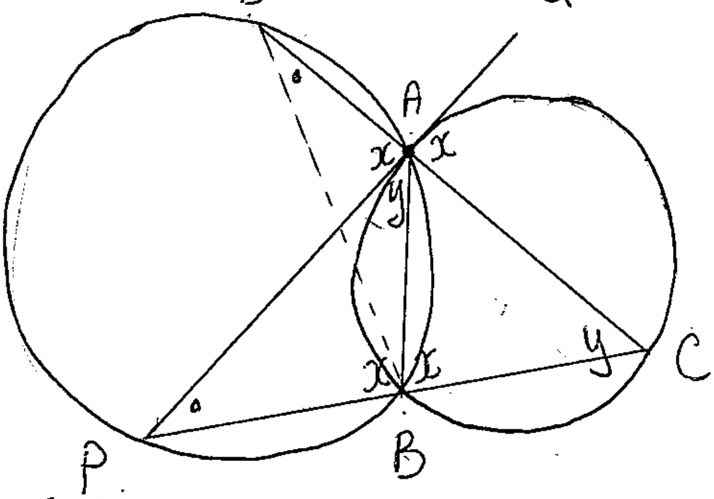
$$= 2^k(2x+k+1)e^{2x}$$

Now $\frac{d^{k+1}}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k}{dx^k}\right)$ ①

$= \frac{d}{dx}\left(2^{k-1}(2x+k)e^{2x}\right)$

PTO.

(e)



data $Q\hat{A}C = x$
$P\hat{A}B = y$

$Q\hat{A}C = D\hat{A}P = x$ vertically opposite angles are equal.
$Q\hat{A}C = A\hat{B}C = x$ alternate segment theorem.
$D\hat{A}P = D\hat{B}P = x$ angles standing on the same arc DP

So $A\hat{B}D = (180 - 2x)$ straight line PBC.

$\therefore$ In $\triangle DBA$, $B\hat{D}A + x + y + 180 - 2x = 180$ angle sum triangle
$B\hat{D}A - x + y = 0$
$B\hat{D}A = -y + x$ or $x - y$.

Very well answered with a variety of approaches.
All very similar
One type of solution shown above.

(3)

Final comment: handwriting often was very difficult to interpret. If I cant read it, I cannot give it full marks.

2015 Extension 1 Mathematics Task 3:
Solutions— Question 10

10. (a) A car braked with a constant deceleration of $16\,\text{m/s}^2$, producing skid marks measuring $200\,\text{m}$ before coming to a stop. How fast was the car travelling when the brakes were first applied? 4

Solution: **Method 1—** Taking u as the initial velocity:

$$\left.\begin{aligned} \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) &= -16, \\ \tfrac{1}{2}v^2 &= -16x + c. \\ \text{When } x &= 200,\ v = 0, \\ \therefore c &= 3200, \\ \text{so } v^2 &= 6400 - 32x. \\ \text{When } x &= 0,\ u^2 = 6400, \\ u &= \pm 80, \\ \text{but } u &> 0, \\ \text{thus } u &= 80\,\text{m/s}. \end{aligned}\right\} \text{ or } \left\{\begin{aligned} v\frac{dv}{dx} &= -16, \\ \int_u^0 v.dv &= \int_0^{200} -16.dx, \\ \left[\frac{v^2}{2}\right]_u^0 &= \Big[-16x\Big]_0^{200}, \\ 0 - \frac{u^2}{2} &= -3200 + 0, \\ u^2 &= 6400, \\ \therefore u &= 80\,\text{m/s as } u > 0. \end{aligned}\right.$$

Comment: Candidates who used this method were generally successful, although placement of limits in the right-hand version caused some confusion. When taking square roots, candidates should explain why the positive or negative was chosen.

Method 2—

$$\begin{aligned} \ddot{x} &= -16, \\ \dot{x} &= -16t + c \ldots\ldots\ldots\ldots \boxed{1} \\ x &= -8t^2 + ct + d \ldots\ldots\ldots \boxed{2} \\ \text{When } t &= 0,\ x = 0 \Rightarrow d = 0, \\ \therefore x &= -8t^2 + ct. \\ \text{When } x &= 200,\ v = 0,\ t = T \text{ say}, \\ 0 &= -16T + c, \text{ from } \boxed{1} \\ \text{so } c &= 16T, \\ \text{and } 200 &= -8T^2 + 16T.T, \text{ from } \boxed{2} \\ &= 8T^2, \\ T^2 &= 25, \\ T &= \pm 5, \\ \text{but } T \geqslant\ & 0 \Rightarrow T = 5\,\text{s}, \\ \therefore c &= 16 \times 5, \\ &= 80. \\ \text{When } t &= 0,\ \dot{x} = 80, \end{aligned}$$

i.e. the initial speed is $80\,\text{m/s}$.

Comment: Many candidates got lost in the algebra, often failing to make the distinction between the variable t and the constant T.

Solutions continue overleaf—

Solution: (Continued)

Method 3— Using the equation $v^2 = u^2 + 2aS$ from Physics,

and substituting known values:
$$\begin{aligned} 0 &= u^2 + 2(-16) \times 200, \\ u^2 &= 6400, \\ u &= \pm 80, \\ \text{but } u &\geqslant 0, \\ \text{thus } u &= 80\,\text{m/s}. \end{aligned}$$

Comment: This method garnered at most 2 marks because the equation was simply stated rather than derived. See below for an example of how the equation (marked as $\boxed{2}$) could have been derived.

Method 4—
$$\begin{aligned} \frac{dv}{dt} &= -16. \\ \therefore v &= -16t + c. \\ \text{When } t &= 0,\ v = c = v_0,\ \text{say}, \\ \text{so } v &= \frac{dx}{dt} = -16t + v_0 \ldots\ldots\ldots\ldots\ldots \boxed{1} \\ \therefore x &= -8t^2 + v_0 t + c. \\ \text{When } t &= 0,\ x = 0 \Rightarrow c = 0. \\ \text{and } x &= v_0 t - 8t^2. \\ \text{But } t &= \frac{v_0 - v}{16},\ \text{from } \boxed{1} \\ \therefore x &= \frac{v_0^2 - v v_0}{16} - \frac{(v_0 - v)^2}{32}, \\ &= \frac{2v_0^2 - 2v v_0 - v_0^2 + 2v v_0 - v^2}{32}, \\ 32x &= v_0^2 - v^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \boxed{2} \\ \text{When } x &= 200,\ v = 0, \\ \therefore v_0 &= \pm 80, \\ \text{but } v_0 &\geqslant 0, \end{aligned}$$

so the initial velocity, v_0, is 80 m/s.

(b) Consider the function $f(x) = 3x\tan^{-1}(2x)$.

(i) Write down the range of $f(x)$. 2

Solution: When $x > 0,\ 3x > 0,\ \tan^{-1}(2x) > 0,$
so $f(x) > 0.$
When $x < 0,\ 3x < 0,\ \tan^{-1}(2x) < 0,$
so $f(x) > 0.$
Also $f(0) = 0.$
Hence range: $f(x) \geqslant 0.$

Comment: Many candidates seemed to confuse domain and range. Others failed to see that the function was even. Very few gained full (or any) marks.

(ii) Show that $f'(x) = 3\tan^{-1}(2x) + \dfrac{6x}{1+4x^2}$. 2

Solution:
$$f'(x) = 3 \times \tan^{-1}(2x) + 3x \times 2 \times \frac{1}{1+(2x)^2},$$
(using both the product and chain rules)
$$= 3\tan^{-1}(2x) + \frac{6x}{1+4x^2}.$$

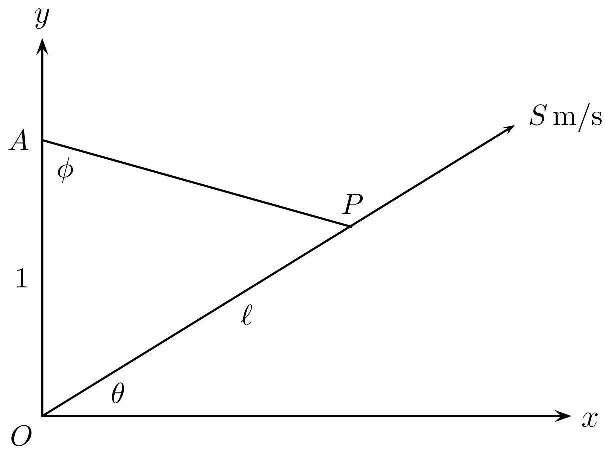
Comment: This was generally well done, although some candidates do not realise that show means give an idea of how the result was obtained. Simply affirming that you agree by rewriting the statement is insufficient.

(iii) Hence, evaluate the area enclosed by the $g(x) = \tan^{-1}(2x)$, the x-axis, and the lines $x = \dfrac{1}{2}$ and $x = \dfrac{\sqrt{3}}{2}$. 3

Solution:
$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} f'(x).dx = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} 3\tan^{-1}(2x).dx + \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{6x.dx}{1+4x^2},$$
$$3\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \tan^{-1}(2x).dx = \Big[f(x)\Big]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} - \frac{3}{4}\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{8x.dx}{1+4x^2},$$
$$\therefore \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \tan^{-1}(2x).dx = \frac{1}{3}\left\{\frac{3\sqrt{3}}{2}.\frac{\pi}{3} - \frac{3}{2}.\frac{\pi}{4}\right\} - \frac{1}{4}\Big[\ln\left(1+4x^2\right)\Big]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}},$$
$$= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{8} - \frac{1}{4}\ln\left(\frac{1+4\times\frac{3}{4}}{1+4\times\frac{1}{4}}\right),$$
$$= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{8} - \frac{1}{4}\ln 2.$$

Comment: "Hence" means *hence*, *i.e.* your method must follow directly from the previous result. This is not the same as "*Hence or otherwise*" when other methods (like parts) with no connexion to earlier results are permitted.

(c) A point A lies on the y-axis, 1 unit from the origin in the positive direction. Another point P lies on the line $y = x \tan\theta$, where $0^\circ \leqslant \theta \leqslant 90^\circ$, and travels along the line with some speed S m/s. Let ℓ be the distance from P to the origin, and let $\angle OAP$ be ϕ.



(i) Prove that $\ell = \dfrac{\sin\phi}{\cos(\phi - \theta)}$. 2

Solution:

$$\begin{aligned} \angle AOP &= 90^\circ - \theta \text{ (adjacent complementary } \angle\text{s)}, \\ \angle APO + \phi + 90^\circ - \theta &= 180^\circ \text{ (angle sum } \triangle AOP), \\ \therefore \angle APO &= 90^\circ - (\phi - \theta). \\ \frac{\ell}{\sin\phi} &= \frac{1}{\sin\big(90^\circ - (\phi - \theta)\big)} \text{ (sine rule)}, \\ \therefore \ell &= \frac{\sin\phi}{\cos(\phi - \theta)}. \end{aligned}$$

Comment: Some candidates omitted showing clearly how they arrived at a value for $\angle APO$.

(ii) Show that when $\phi = 2\theta$,

(α) $\dot{\phi} = S\cos\theta$. [3]

Solution:
$$\frac{d\ell}{d\phi} = \frac{\cos(\phi-\theta).\cos\phi - \big(-\sin(\phi-\theta).\sin\phi\big)}{\cos^2(\phi-\theta)},$$
$$= \frac{\cos\big((\phi-\theta)-\phi\big)}{\cos^2(\phi-\theta)},$$
$$= \frac{\cos(-\theta)}{\cos^2(\phi-\theta)}.$$
$$\therefore \frac{d\phi}{d\ell} = \frac{\cos^2(\phi-\theta)}{\cos\theta} \text{ and } \frac{d\ell}{dt} = S,$$
$$\text{so } \frac{d\phi}{dt} = \frac{S.\cos^2(\phi-\theta)}{\cos\theta}.$$
$$\text{And when } \phi = 2\theta,\ \frac{d\phi}{dt} = \frac{S.\cos^2\theta}{\cos\theta},$$
$$i.e.\ \dot{\phi} = S\cos\theta.$$

Comment: A worrying number of candidates did not seem to understand the notation of Newtonian fluxions,
$$i.e. \text{ that } \dot{\phi} \Rightarrow \frac{d\phi}{dt};$$
they apparently assumed that $\dot{\phi}$ was the same as ϕ'. Many failed to realise that θ is a constant and at the instant when $\phi = 2\theta$, the values are momentarily fixed, so all the calculus should be done before considering that moment.

(β) $\ddot{\phi} = -\dot{\phi} \times S \times \ell$. [2]

Solution:
$$\frac{d}{dt}\Big(\frac{d\phi}{dt}\Big) = \frac{S \times 2 \times \big(-\sin(\phi-\theta)\big) \times \cos(\phi-\theta) \times \dot{\phi}}{\cos\theta},$$
$$= \frac{-\dot{\phi} \times S \times 2\sin(\phi-\theta) \times \cos(\phi-\theta)}{\cos\theta}.$$
$$\text{So when } \phi = 2\theta,\ \ell = \frac{\sin 2\theta}{\cos\theta} \ \big(\text{from part (i)}\big),$$
$$= \frac{2\sin\theta\cos\theta}{\cos\theta},$$
$$= 2\sin\theta,$$
$$\text{and } \ddot{\phi} = \frac{-\dot{\phi} \times S \times 2\sin\theta \times \cos\theta}{\cos\theta}.$$
$$= -\dot{\phi} \times S \times 2\sin\theta,$$
$$= -\dot{\phi} \times S \times \ell.$$

Comment: Candidates had the same kinds of problems in this part as in the previous one.